Solution to HW9

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§16.5

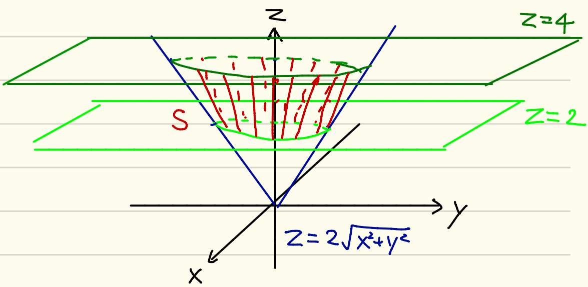
Finding Parametrizations

In Exercises 1–16, find a parametrization of the surface. (There are many correct ways to do these, so your answers may not be the same as those in the back of the book.)

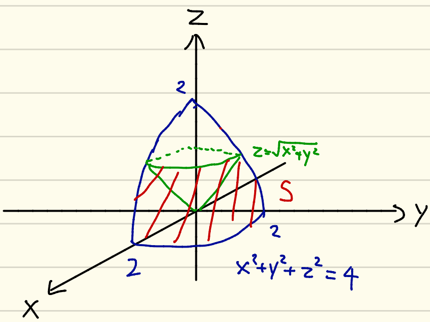
4. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$

6. **Spherical cap** The portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant between the xy-plane and the cone $z = \sqrt{x^2 + y^2}$

Sol) (4) $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + 2r\,\vec{k}$, $1 \le r \le 2$, $0 \le \theta < 2\pi$



(6) $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + \sqrt{4-r^2}\,\vec{k}$, $\sqrt{2} \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$



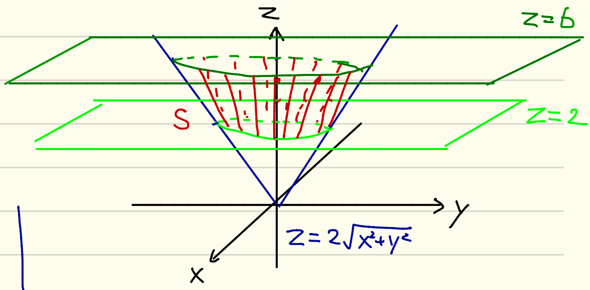
Surface Area of Parametrized Surfaces

In Exercises 17–26, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are many correct ways to set up the integrals, so your integrals may not be the same as those in the back of the book. They should have the same values, however.)

19. Cone frustum The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$

25. Sawed-off sphere The lower portion cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$

Sol) (19) $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + 2r\,\vec{k}$, $1 \le r \le 3$, $0 \le \theta < 2\pi$



$\vec{r}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + 2\vec{k}$

$\vec{r}_\theta = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

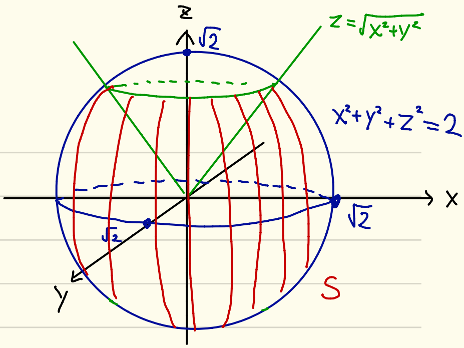
$$= -2r\cos\theta\,\vec{i} - 2r\sin\theta\,\vec{j} + r\,\vec{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{(-2r\cos\theta)^2 + (-2r\sin\theta)^2 + r^2} = \sqrt{5}\,r$$

$$\therefore \text{Area}(S) = \int_0^{2\pi}\int_1^3 \sqrt{5}\,r\,dr\,d\theta = 2\pi\cdot\left[\frac{\sqrt{5}r^2}{2}\right]_1^3 = 8\sqrt{5}\pi$$

25) $\vec{r}(\phi, \theta) = \sqrt{2}\sin\phi\cos\theta\,\vec{i}$
$+ \sqrt{2}\sin\phi\sin\theta\,\vec{j}$
$+ \sqrt{2}\cos\phi\,\vec{k}$

where $\frac{\pi}{4} \leq \phi \leq \pi$; $0 \leq \theta < 2\pi$



$\vec{r}_\phi = \sqrt{2}\cos\phi\cos\theta\,\vec{i} + \sqrt{2}\cos\phi\sin\theta\,\vec{j} - \sqrt{2}\sin\phi\,\vec{k}$

$\vec{r}_\theta = -\sqrt{2}\sin\phi\sin\theta\,\vec{i} + \sqrt{2}\sin\phi\cos\theta\,\vec{j}$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2}\cos\phi\cos\theta & \sqrt{2}\cos\phi\sin\theta & -\sqrt{2}\sin\phi \\ -\sqrt{2}\sin\phi\sin\theta & \sqrt{2}\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= 2\sin^2\phi\cos\theta\,\vec{i} + 2\sin^2\phi\sin\theta\,\vec{j} + 2\cos\phi\sin\phi\,\vec{k}$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{(2\sin^2\phi\cos\theta)^2 + (2\sin^2\phi\sin\theta)^2 + (2\cos\phi\sin\phi)^2} = 2\sin\phi$$

$$\therefore \text{Area}(S) = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi} 2\sin\phi\,d\phi\,d\theta = 2\pi\left[-2\cos\phi\right]_{\frac{\pi}{4}}^{\pi} = -4\pi\left(-1-\frac{\sqrt{2}}{2}\right)$$

$$= (4+2\sqrt{2})\pi$$

Planes Tangent to Parametrized Surfaces

The tangent plane at a point $P_0(f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is the plane through P_0 normal to the vector $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$, the cross product of the tangent vectors $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ at P_0. In Exercises 27–30, find an equation for the plane tangent to the surface at P_0. Then find a Cartesian equation for the surface and sketch the surface and tangent plane together.

30. Parabolic cylinder The parabolic cylinder surface $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, $-\infty < x < \infty$, $-\infty < y < \infty$, at the point $P_0(1, 2, -1)$ corresponding to $(x, y) = (1, 2)$

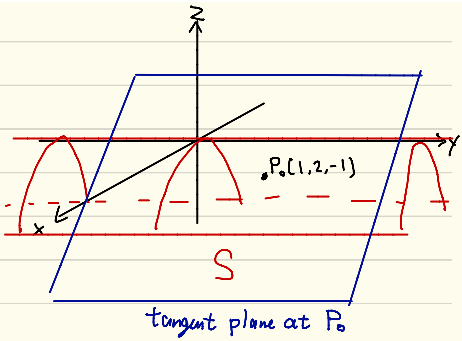
Sol) $\vec{r}(x,y) = x\vec{i} + y\vec{j} - x^2\vec{k}$

$\vec{r}_x = \vec{i} - 2x\vec{k}$; $\vec{r}_x(P_0) = \vec{i} - 2\vec{k}$

$\vec{r}_y = \vec{j}$; $\vec{r}_y(P_0) = \vec{j}$

$$(\vec{r}_x \times \vec{r}_y)(P_0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2\vec{i} + \vec{k}$$



$\therefore$ Equation of tangent plane : $((x-1)\vec{i} + (y-2)\vec{j} + (z+1)\vec{k}) \cdot (2\vec{i} + \vec{k}) = 0$

i.e. $2x + z = 1$.

Equation of surface : $z = -x^2$.

33. a. Parametrization of an ellipsoid Recall the parametrization $x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta \le 2\pi$ for the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ (Section 3.9, Example 5). Using the angles θ and ϕ in spherical coordinates, show that

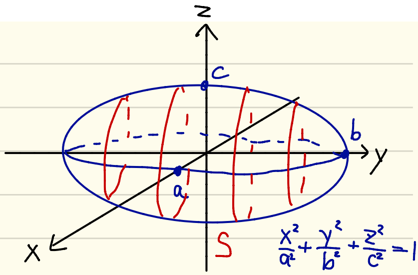
$$\mathbf{r}(\theta, \phi) = (a\cos\theta\cos\phi)\mathbf{i} + (b\sin\theta\cos\phi)\mathbf{j} + (c\sin\phi)\mathbf{k}$$

is a parametrization of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

b. Write an integral for the surface area of the ellipsoid, but do not evaluate the integral.

Sol) (b) $\vec{r}(\phi, \theta) = a\sin\phi\cos\theta\,\vec{i} + b\sin\phi\sin\theta\,\vec{j} + c\cos\phi\,\vec{k}$

where $0 \le \phi \le \pi$; $0 \le \theta < 2\pi$



$\vec{r}_\phi = a\cos\phi\cos\theta\,\vec{i} + b\cos\phi\sin\theta\,\vec{j} - c\sin\phi\,\vec{k}$

$\vec{r}_\theta = -a\sin\phi\sin\theta\,\vec{i} + b\sin\phi\cos\theta\,\vec{j}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\phi\cos\theta & b\cos\phi\sin\theta & -c\sin\phi \\ -a\sin\phi\sin\theta & b\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= bc\sin^2\phi\cos\theta\,\vec{i} + ac\sin^2\phi\sin\theta\,\vec{j} + ab\cos\phi\sin\phi\,\vec{k}$$

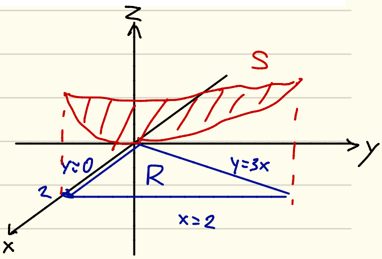
$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{(bc\sin^2\phi\cos\theta)^2 + (ac\sin^2\phi\sin\theta)^2 + (ab\cos\phi\sin\phi)^2}$$

$$= \sqrt{b^2c^2\sin^4\phi\cos^2\theta + a^2c^2\sin^4\phi\sin^2\theta + a^2b^2\cos^2\phi\sin^2\phi}$$

$$\therefore \text{Area}(S) = \int_0^{2\pi}\int_0^{\pi} \sqrt{b^2c^2\sin^4\phi\cos^2\theta + a^2c^2\sin^4\phi\sin^2\theta + a^2b^2\cos^2\phi\sin^2\phi}\; d\phi\, d\theta$$

41. Find the area of the surface $x^2 - 2y - 2z = 0$ that lies above the triangle bounded by the lines $x = 2$, $y = 0$, and $y = 3x$ in the xy-plane.

Sol) $f(x,y,z) = x^2 - 2y - 2z$;



$\nabla f(x,y,z) = 2x\vec{i} - 2\vec{j} - 2\vec{k}$;

$|\nabla f(x,y,z)| = \sqrt{(2x)^2 + (-2)^2 + (-2)^2} = 2\sqrt{x^2+2}$

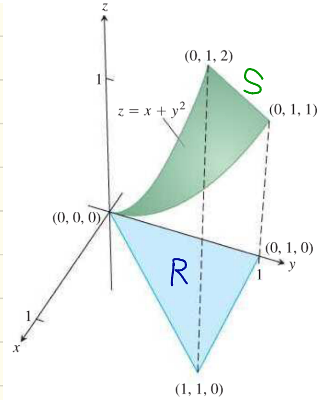
$\nabla f(x,y,z)\cdot\vec{k} = -2$; $|\nabla f(x,y,z)\cdot\vec{k}| = 2$

$\therefore \text{Area}(S) = \iint_R \frac{2\sqrt{x^2+2}}{2}\, dA = \int_0^2 \int_0^{3x} \sqrt{x^2+2}\, dy\,dx$

$= \int_0^2 3x\sqrt{x^2+2}\, dx = \frac{3}{2}\left[\frac{(x^2+2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = 6\sqrt{6} - 2\sqrt{2}$

§16.6

15. Integrate $G(x, y, z) = z - x$ over the portion of the graph of $z = x + y^2$ above the triangle in the xy-plane having vertices $(0, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 0)$. (See accompanying figure.)



Sol $f(x,y,z) = x + y^2 - z$;

$\nabla f(x,y,z) = \vec{i} + 2y\vec{j} - \vec{k}$;

$|\nabla f(x,y,z)| = \sqrt{(1)^2 + (2y)^2 + (1)^2} = \sqrt{4y^2+2}$

$\nabla f(x,y,z) \cdot \vec{k} = -1$; $|\nabla f(x,y,z) \cdot \vec{k}| = 1$

$$\therefore \iint_S G \cdot d\sigma = \iint_R (x + y^2 - x) \cdot \sqrt{4y^2+2}\, dA$$

$$= \int_0^1 \int_0^y y^2 \sqrt{4y^2+2}\, dx\, dy = \int_0^1 y^3 \sqrt{4y^2+2}\, dy$$

$$= \frac{2}{3} \cdot \frac{1}{8} \int_0^1 y^2\, d\left((4y^2+2)^{\frac{3}{2}}\right)$$

$$= \frac{1}{12}\left(\left[y^2 (4y^2+2)^{\frac{3}{2}}\right]_0^1 - \int_0^1 (4y^2+2)^{\frac{3}{2}} \cdot 2y\, dy\right)$$

$$= \frac{1}{12}\left(6\sqrt{6} - \frac{2}{5} \cdot \frac{1}{4}\left[(4y^2+2)^{\frac{5}{2}}\right]_0^1\right)$$

$$= \frac{1}{12}\left(6\sqrt{6} - \frac{1}{10} \cdot (36\sqrt{6} - 4\sqrt{2})\right)$$

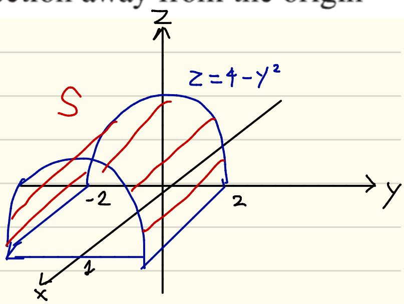
$$= \frac{1}{30}\left(15\sqrt{6} - 9\sqrt{6} + \sqrt{2}\right) = \frac{1}{30}\left(6\sqrt{6} + \sqrt{2}\right)$$

Finding Flux Across a Surface

In Exercises 19–28, use a parametrization to find the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ across the surface in the given direction.

19. Parabolic cylinder $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$

21. Sphere $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin



Sol) 19) $\vec{r}(x,y) = x\vec{i} + y\vec{j} + (4-y^2)\vec{k}$

$\vec{r}_x = \vec{i}$; $\vec{r}_y = \vec{j} - 2y\vec{k}$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\vec{j} + \vec{k} \;;$$

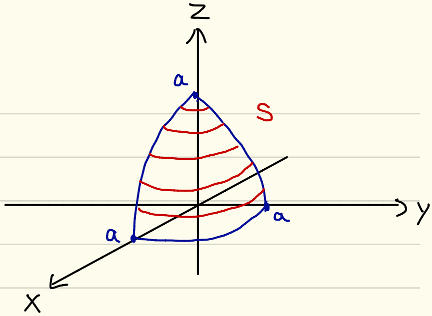
$$\vec{F}\cdot\vec{n}\,d\sigma = (z^2\vec{i} + x\vec{j} - 3z\vec{k})\cdot(2y\vec{j} + \vec{k})\,dy\,dx$$

$$= (2xy - 3z)\,dy\,dx = (2xy - 3(4-y^2))\,dy\,dx = (3y^2 + 2xy - 12)\,dy\,dx$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \int_0^1\int_{-2}^{2}(3y^2 + 2xy - 12)\,dy\,dx$$

$$= \int_0^1 \left[y^3 + xy^2 - 12y\right]_{-2}^{2} dx$$

$$= \int_0^1 (-32)\,dx = -32$$



21) $\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$

where $0 \le \phi \le \frac{\pi}{2}$; $0 \le \theta \le \frac{\pi}{2}$

$$\vec{r}_\phi = a\cos\phi\cos\theta\,\vec{i} + a\cos\phi\sin\theta\,\vec{j} - a\sin\phi\,\vec{k}$$

$$\vec{r}_\theta = -a\sin\phi\sin\theta\,\vec{i} + a\sin\phi\cos\theta\,\vec{j}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\vec{i} + a^2\sin^2\phi\sin\theta\,\vec{j} + a^2\cos\phi\sin\phi\,\vec{k}$$

$$\vec{F}\cdot\vec{n}\,d\sigma = (a\cos\phi\,\vec{k})\cdot(a^2\sin^2\phi\cos\theta\,\vec{i} + a^2\sin^2\phi\sin\theta\,\vec{j} + a^2\cos\phi\sin\phi\,\vec{k})\,d\phi\,d\theta$$

$$= a^3\cos^2\phi\sin\phi\,d\phi\,d\theta$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} a^3\cos^2\phi\sin\phi\,d\phi\,d\theta$$

$$= a^3\cdot\frac{\pi}{2}\cdot\left[-\frac{\cos^3\phi}{3}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{6}a^3$$